Nice Banach Modules and Invariant Subspaces

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Abstract

Let $\mathbb{A}$ be a semisimple unital commutative Banach algebra. We say that a Banach $\mathbb{A}$-module M is *nice* if every proper closed submodule of M is contained in a closed submodule of M of codimension 1. We provide examples of nice and non-nice modules.

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1 Introduction

In this article, all vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers. As usual, $\mathbb{R}$ is the field of real numbers, $\mathbb{N}$ is the set of all positive integers, $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_+$ is the set of non-negative integers. For a Banach space X, $L(X)$ stands for the algebra of bounded linear operators on X, while X^* is the space of continuous linear functionals on X. For $T \in L(X)$, its dual is denoted T^*: $T^* \in L(X^*)$, $T'f(x) = f(Tx)$ for every $f \in X^*$ and every $x \in X$.

Throughout this article $\mathbb{A}$ stands for a unital commutative semisimple Banach algebra. It is well-known and is a straightforward application of the Gelfand theory [2, 1] that for an ideal J in $\mathbb{A}$,

$$J = \mathbb{A} \iff J \text{ is dense in } \mathbb{A} \iff \varkappa\big|_J \neq 0 \text{ for every } \varkappa \in \Omega(\mathbb{A}),$$

where $\Omega(\mathbb{A})$ is the spectrum of $\mathbb{A}$, that is, $\Omega(\mathbb{A})$ is the set of all (automatically continuous) non-zero algebra homomorphisms from $\mathbb{A}$ to $\mathbb{C}$ (endowed with the $*$-weak topology). Equivalently, every proper ideal in $\mathbb{A}$ is contained in a closed ideal of codimension 1.

Let $\Omega^+(\mathbb{A})$ be the set of all algebra homomorphisms from $\mathbb{A}$ to $\mathbb{C}$. That is, $\Omega^+(\mathbb{A})$ is $\Omega(\mathbb{A})$ together with the identically zero map from $\mathbb{A}$ to $\mathbb{C}$. The main purpose of this paper is to draw attention to possible extensions of the above fact to Banach $\mathbb{A}$-modules. Clearly, each $\varkappa \in \Omega^+(\mathbb{A})$ gives rise to the 1-dimensional $\mathbb{A}$-module $\mathbb{C}_\varkappa$ being $\mathbb{C}$ with the $\mathbb{A}$-module structure given by the multiplication $a\lambda = \varkappa(a)\lambda$ for every $a \in \mathbb{A}$ and $\lambda \in \mathbb{C}$. It is also rather obvious that we have just listed all the 1-dimensional $\mathbb{A}$-modules up to an isomorphism.

Definition 1.1. Let M be a Banach $\mathbb{A}$-module. A *character* on M is a non-zero $\varphi \in M^*$ such that there exists $\varkappa \in \Omega^+(\mathbb{A})$ making φ into an $\mathbb{A}$-module morphism from M to $\mathbb{C}_\varkappa$.

Obviously, the kernel of a character on a Banach $\mathbb{A}$-module M is a closed $\mathbb{A}$-submodule of M.

Definition 1.2. Let M be a Banach $\mathbb{A}$-module. We say that M is *nice* if for every proper closed submodule of M is contained in a closed submodule of codimension 1. Equivalently, M is nice if and only if for every proper closed submodule N of M, there is a character φ on M such that φ vanishes on N.

The general question we would like to raise is:

Question 1.3. *Characterize nice Banach $\mathbb{A}$-modules.*

The remark we started with ensures that $\mathbb{A}$ is nice as an $\mathbb{A}$-module. In this paper we just present examples of nice and non-nice modules. Before even formulating the results, I would like to put forth my personal motivation for even looking at this question. Assume for a minute that $\mathbb{A}$ is a subalgebra of $L(X)$ for some Banach space X. We allow the norm topology of $\mathbb{A}$ to be stronger (not necessarily strictly) than

the topology defined by the norm inherited from $L(X)$. The multiplication $(A, x) \mapsto Ax$ defines a Banach $\mathbb{A}$-module structure on X. What are the characters on X? Why, one easily sees that they are exactly the common eigenvectors of A^* for $A \in \mathbb{A}$. What are the $\mathbb{A}$-submodules of X? They are exactly the invariant subspaces for the action of $\mathbb{A}$ on X. Thus the $\mathbb{A}$-module X is nice exactly when every non-trivial closed $\mathbb{A}$-invariant subspace of X is contained in a closed $\mathbb{A}$-invariant hyperplane. Thus X being a nice $\mathbb{A}$-module translates into a strong and important property of the lattice of $\mathbb{A}$-invariant subspaces. Note that under relatively mild extra assumptions on $\mathbb{A}$, the nicety of X results in every closed $\mathbb{A}$-invariant subspace being the intersection of a collection of characters on X thus providing a complete description of the lattice of $\mathbb{A}$. A byproduct of this observation is the following easy example of a non-nice module.

Example 1.4. *Let Ω be a non-empty compact subset of $\mathbb{C}$ with no isolated points and μ be a finite σ-additive purely non-atomic Borel measure on $\mathbb{C}$, whose support is exactly Ω. The pointwise multiplication equips $L^2(\Omega, \mu)$ with the structure of a Banach $C(\Omega)$-module. This module is non-nice.*

Proof. The $C(\Omega)$-module $L^2(\Omega, \mu)$ does have plenty of closed submodules. For instance, every Borel subset A of Ω satisfying $\mu(A) \neq 0$ and $\mu(\Omega \backslash A) \neq 0$ generates a closed non-trivial submodule $M_A = \{f \in L^2(\Omega, \mu) : f \text{ vanishes outside } A\}$. On the other hand, we can always pick $f \in C(\Omega)$ satisfying $\mu(f^{-1}(\lambda)) = 0$ for every $\lambda \in \mathbb{C}$. In this case the dual of the multiplication by f operator on $L^2(\Omega, \mu)$ has empty point spectrum. Due to the above remark, our module possesses no characters at all (while possessing non-trivial closed submodules) and therefore can not possibly be nice. □

In the positive direction we have the following two rather easy statements.

Proposition 1.5. *The finitely generated free $\mathbb{A}$-module $\mathbb{A}^n$ is nice.*

Proposition 1.6. *Let Ω be a Hausdorff compact topological space and X be a Banach space. Then the $C(\Omega)$-module $C(\Omega, X)$ is nice, where $C(\Omega, X)$ carries the natural norm $\|f\| = \sup\{\|f(\omega)\|_X : \omega \in \Omega\}$ and the module structure is given by the pointwise multiplication.*

Note that Example 1.4 is rather cheatish since the non-nicety comes from the lack of characters. A really interesting situation is when a non-nice module possesses a separating set of characters. The following result says that this is quite possible. Recall that the Sobolev space $W^{1,2}[0,1]$ consists of the functions $f : [0,1] \to \mathbb{C}$ absolutely continuous on any bounded subinterval of I and such that $f' \in L_2[0,1]$. The space $W^{1,2}[0,1]$ with the inner product

$$\langle f, g\rangle_{1,2} = \int_0^1 (f(t)\overline{g(t)} + f'(t)\overline{g'(t)})\, dt$$

is a separable Hilbert space. We denote $\|f\|_{1,2} = \sqrt{\langle f, f\rangle_{1,2}}$. Apart from being a Hilbert space, $W^{1,2}[0,1]$ is also a Banach algebra with respect to the pointwise multiplication (if one strives for the submultiplicativity of the norm together with the identity $\|1\| = 1$, he or she has to pass to an equivalent norm).

We say that a function f defined on $[0,1]$ and taking values in a Banach space X is *absolutely continuous* if there exists an (automatically unique up to a Lebesgue-null set) Borel measurable function $g : [0,1] \to X$ such that

$$\int_0^1 \|g(t)\|\, dt < +\infty \quad \text{and} \quad \int_0^x g(t)\, dt = f(x) \quad \text{for each} \quad x \in [0,1],$$

where the second integral is considered in the Bochner sense. We denote the function g as f'. If H is a Hilbert space. The symbol $W^{1,2}([0,1], H)$ stands for the space of absolutely continuous functions $f : [0,1] \to H$ such that

$$\int_0^1 \|f'(t)\|^2\, dt < +\infty.$$

The space $W^{1,2}([0,1], H)$ with the inner product

$$\langle f, g\rangle = \int_0^1 (\langle f(t), g(t)\rangle_H + \langle f'(t), g'(t)\rangle_H)\, dt$$

is a Hilbert space and is separable if H is separable. In any case if $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis of H, then the space $W^{1,2}([0,1],H)$ is naturally identified with the Hilbert direct sum of $|A|$ copies of $W^{1,2}[0,1]$: $f\mapsto\{f_\alpha\}_{\alpha\in A}$, where $f_\alpha(t)=\langle f(t),e_\alpha\rangle_H$. It is also clear that $W^{1,2}([0,1],H)$ is naturally isomorphic to the Hilbert space tensor product of $W^{1,2}[0,1]$ and H. Clearly, $W^{1,2}([0,1],H)$ is a Banach $W^{1,2}[0,1]$-module. This module possesses a lot of characters. Indeed, if $t\in[0,1]$ and $x\in H$, then the functional $f\mapsto\langle f(t),x\rangle_H$ is a character on $W^{1,2}([0,1],H)$. Moreover, these characters do separate points of $W^{1,2}([0,1],H)$.

Theorem 1.7. *Let H be a Hilbert space. Then the $W^{1,2}[0,1]$-module $W^{1,2}([0,1],H)$ is nice if and only if H is finite dimensional.*

2 Proof of Proposition 1.6

It is easy to see that a character on $C(\Omega,X)$ is exactly a functional of the form

$$\varkappa_{\omega,\varphi}(f)=\varphi(f(\omega)),\quad \text{where } \omega\in\Omega \text{ and } \varphi\in X^*\setminus\{0\}. \tag{2.1}$$

The following lemma describes all closed submodules of $C(\Omega,X)$.

Lemma 2.1. *Let M be a $C(\Omega)$-submodule of $C(\Omega,X)$ and for each $\omega\in\Omega$ let $M_\omega=\{f(\omega): f\in M\}$. Then the closure $\overline{M}$ of M in $C(\Omega,X)$ satisfies*

$$\overline{M}=\widetilde{M},\quad \textit{where } \widetilde{M}=\{f\in C(\Omega,X): f(\omega)\in\overline{M}_\omega \ \textit{ for each } \omega\in\Omega\}, \tag{2.2}$$

with $\overline{M}_\omega$ being the closure in X of M_ω.

Proof. Since $M\subseteq\widetilde{M}$ and $\widetilde{M}$ is closed, we have $\overline{M}\subseteq\widetilde{M}$. Let $f\in\widetilde{M}$ and $\varepsilon>0$. The desired equality will be verified if we show that there is $g\in M$ such that $\|f-g\|<\varepsilon$. Indeed, in this case $\widetilde{M}\subseteq\overline{M}$ and therefore $\overline{M}=\widetilde{M}$.

Take $\omega\in\Omega$. Since M_ω is dense in $\overline{M}_\omega$, there is $g_\omega\in M$ such that $\|f(\omega)-g_\omega(\omega)\|_X<\varepsilon$. Then $V_\omega=\{s\in\Omega:\|f(s)-g_\omega(s)\|_X<\varepsilon\}$ is an open subset of Ω containing ω. Thus $\{V_\omega\}_{\omega\in\Omega}$ is an open covering of Ω. Since for every open covering of a Hausdorff compact topological space, there is a finite partition of unity consisting of continuous functions and subordinate to the covering [4], there are $\omega_1,\dots,\omega_n\in\Omega$ and $\rho_1,\dots,\rho_n\in C(\Omega)$ such that

$$\begin{aligned}&0\leqslant\rho_j(s)\leqslant 1 \ \text{ for every } 1\leqslant j\leqslant n \text{ and } s\in\Omega;\\ &\rho_j(s)=0 \ \text{ whenever } 1\leqslant j\leqslant n \text{ and } s\in\Omega\setminus V_{\omega_j};\\ &\rho_1(s)+\ldots+\rho_n(s)=1 \ \text{ for each } s\in\Omega.\end{aligned} \tag{2.3}$$

Now we set $g=\rho_1g_{\omega_1}+\ldots+\rho_ng_{\omega_n}$. Since M is a $C(\Omega)$-module and $g_\omega\in M$, we have $g\in M$. Using (2.3) together with the inequality $\|f(s)-g_{\omega_j}(s)\|_X<\varepsilon$ for $s\in V_{\omega_j}$, we easily see that $\|f(s)-g(s)\|_X<\varepsilon$ for each $s\in\Omega$. Hence $g\in M$ and $\|f-g\|<\varepsilon$, which completes the proof. ☐

We are ready to prove Proposition 1.6. Let M be a closed submodule of $C(\Omega,X)$ such that none of the characters on $C(\Omega,X)$ vanishes on M. According to (2.1), the latter means that every $M_\omega=\{f(\omega): f\in M\}$ is dense in X and therefore $\overline{M}_\omega=X$ for each $\omega\in\Omega$. Since M is closed, Lemma 2.1 says that $M=C(\Omega,X)$. The proof is complete.

3 Proof of Propositions 1.5

We start with the following easy observation. Let $\varkappa\in\Omega(\mathbb{A})$. Then the $\mathbb{A}$-module morphisms $\psi:\mathbb{A}^n\to\mathbb{C}_\varkappa$ are all given by

$$\varphi_c(a_1,\dots,a_n)=\sum_{j=1}^n c_j\varkappa(a_j),\quad \text{where } c\in\mathbb{C}^n.$$

We shall prove a statement slightly stronger than Proposition 1.5.

Proposition 3.1. *Let $n \in \mathbb{N}$ and M be an $\mathbb{A}$-submodule of the free $\mathbb{A}$-module $\mathbb{A}^n$. Assume also that none of the characters on $\mathbb{A}^n$ vanishes on M. Then $M = \mathbb{A}^n$.*

Proof. We use induction with respect to n. The case $n = 1$ is trivial (see the remark at the very start of the article). Assume now that $n \geqslant 2$ and that the conclusion of Proposition 1.5 holds for every smaller n. We interpret $\mathbb{A}^n$ as $\mathbb{A}^n = \mathbb{A} \times \mathbb{A}^{n-1}$. The induction hypothesis easily implies that that the projection of M onto $\mathbb{A}^{n-1}$ is onto. Let $J \subseteq \mathbb{A}$ be defined by $M \cap (\mathbb{A} \times \{0\}) = J \times \{0\}$. Then J is an ideal in A. If $J = \mathbb{A}$, we can factor out the first component in the product $\mathbb{A} \times \mathbb{A}^{n-1} = \mathbb{A}^n$ and then use the induction hypothesis to conclude that $M = \mathbb{A}^n$. Thus it remains to consider the case $J \neq \mathbb{A}$. Then there is $\varkappa \in \Omega(\mathbb{A})$ such that $J \subseteq \mathtt{ker}\,\varkappa$. Using the definition of J, and the facts that M is an $\mathbb{A}$-module, M projects onto the entire $\mathbb{A}^{n-1}$ and $\varkappa$ vanishes on J, we can define $\psi : \mathbb{A}^{n-1} \to \mathbb{C}$ by the rule $\psi(b) = \varkappa(a)$ if $(a, b) \in M \subseteq \mathbb{A} \times \mathbb{A}^{n-1}$. It is easy to see that ψ is a well-defined continuous linear functional and that $\psi : \mathbb{A}^{n-1} \to \mathbb{C}_\varkappa$ is an $\mathbb{A}$-module morphism. According to the above display there are $c_1, \dots, c_{n-1} \in \mathbb{C}$ such that $\psi(a_1, \dots, a_{n-1}) = \sum\limits_{j=1}^{n-1} c_j \varkappa(a_j)$ for every $a_1, \dots, a_{n-1} \in \mathbb{A}$. By definition of ψ, we now see that $\varphi : \mathbb{A}^n \to \mathbb{C}$ vanishes on M, where φ is defined by the formula $\varphi(a_1, \dots, a_n) = \sum\limits_{j=1}^{n} c_j \varkappa(a_j)$ with $c_n = -1$. By the above display, $\varphi : \mathbb{A}^n \to \mathbb{C}_\varkappa$ is an $\mathbb{A}$-module morphism. Since $c_n \neq 0$, $\varphi \neq 0$ and therefore φ is a character on $\mathbb{A}^n$. We have produced a character on $\mathbb{A}^n$ vanishing on M, which contradicts the assumptions. Thus the case $J \neq \mathbb{A}$ does not occur, which completes the proof. □

4 Proof of Theorem 1.7

In this section, for a function f on an interval I of the real line $\|f\|_2$ will always denote the L^2-norm of f (with respect to the Lebesgue measure), while $\|f\|_\infty$ always stands for the L^∞-norm of f.

Lemma 4.1. *Let $-\infty < \alpha < \beta < +\infty$, $a, b \in \mathbb{C}$ and $\varepsilon > 0$. Then there exists $f \in C^1[\alpha, \beta]$ such that $f(\alpha) = f(\beta) = 0$, $f'(\alpha) = a$, $f'(\beta) = b$ and $\|f\|_\infty < \varepsilon$.*

Proof. Let $\varphi \in C^1[0, \infty)$ be a monotonically non-increasing function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(x) = 0$ for $x \geqslant 1$. For any $\delta \in (0, \frac{\beta-\alpha}{2})$ let

$$f_\delta(x) = \begin{cases} 0 & \text{if } x \in (\alpha + \delta, \beta - \delta), \\ a(x - \alpha)\varphi((x - \alpha)/\delta) & \text{if } x \in [\alpha, \alpha + \delta), \\ b(x - \beta)\varphi((\beta - x)/\delta) & \text{if } x \in (\beta - \delta, \beta]. \end{cases}$$

Obviously, $f_\delta \in C^1[\alpha, \beta]$, $f_\delta(\alpha) = f_\delta(\beta) = 0$, $f'_\delta(\alpha) = a$, $f'_\delta(\beta) = b$ and $\|f\|_\infty \leqslant \delta \max\{|a|, |b|\}$. Hence the function $f = f_\delta$ for $\delta < \varepsilon / \max\{|a|, |b|\}$ satisfies all desired conditions. □

Lemma 4.2. *Let $K \subset [0, 1]$ be a nowhere dense compact set, $a \in C(K)$, $f \in C[0, 1]$ and $\varepsilon > 0$. Then there exists $g \in C^1[0, 1]$ such that $g'\big|_K = a$ and $\|g - f\|_\infty < \varepsilon$.*

Proof. Since $C^1[0, 1]$ is dense in the Banach space $C[0, 1]$, we can, without loss of generality, assume that $f \in C^1[0, 1]$. Since any continuous function on K admits a continuous extension to $[0, 1]$ (one can apply, for instance, the Tietze theorem [4]), there exists $h \in C[0, 1]$ such that $h(x) = a(x) - f'(x)$ for any $x \in K$. Let $\delta > 0$. Since K is nowhere dense, there exist

$$0 = \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n = 1$$

such that $\beta_j - \alpha_j < \varepsilon$ for any $j = 1, \dots, n$ and $K \subset \bigcup\limits_{j=1}^{n} I_j$, where $I_j = [\alpha_j, \beta_j]$. Let

$$a_j = \int_{\alpha_j}^{\beta_j} h(t)\, dt \quad \text{for } 1 \leqslant j \leqslant n - 1.$$

By Lemma 4.1, for $1 \leqslant j \leqslant n-1$, there is $\varphi_j \in C^1[\beta_j, \alpha_{j+1}]$ such that $\varphi_j(\beta_j) = \varphi_j(\alpha_{j+1}) = 0$, $\varphi_j'(\beta_j) = h(\beta_j) + \frac{a_j}{\alpha_{j+1}-\beta_j}$, $\varphi_j'(\alpha_{j+1}) = h(\alpha_{j+1}) + \frac{a_j}{\alpha_{j+1}-\beta_j}$ and $\|\varphi_j\|_\infty < \delta$. Consider the function

$$\psi(x) = \begin{cases} \int_{\alpha_j}^x h(t)\,dt & \text{if } x \in [\alpha_j, \beta_j],\ 1 \leqslant j \leqslant n, \\ \varphi_j(x) + \frac{a_j(x-\alpha_{j+1})}{\beta_j - \alpha_{j+1}} & \text{if } x \in (\beta_j, \alpha_{j+1}),\ 1 \leqslant j \leqslant n-1. \end{cases}$$

The values of φ_j' at β_j and α_{j+1} were chosen in such a way that $\psi \in C^1[0,1]$. Moreover, $\psi'\big|_{I_j} = h$ for $1 \leqslant j \leqslant n$. Hence, $(\psi+f)'\big|_K = a$. Let us estimate $\|\psi\|_\infty$. If $1 \leqslant j \leqslant n-1$ and $x \in [\beta_j, \alpha_{j+1}]$, then $|\psi(x)| \leqslant \delta + |a_j| \leqslant \delta + |\beta_j - \alpha_j|\|h\|_\infty \leqslant \delta(1+\|h\|_\infty)$. If $1 \leqslant j \leqslant n$ and $x \in [\alpha_j, \beta_j]$, then $|\psi(x)| \leqslant |\beta_j - \alpha_j|\|h\|_\infty \leqslant \delta\|h\|_\infty$. Hence $\|\psi\|_\infty \leqslant \delta(1+\|h\|_\infty)$. Choose $\delta < \varepsilon/(1+\|h\|_\infty)$ and denote $g = \psi + f$. Then $g'\big|_K = a$ and $\|g-f\|_\infty = \|\psi\|_\infty < \varepsilon$. □

Lemma 4.3. *Let $K \subset [0,1]$ be a nowhere dense compact set and $\varepsilon > 0$. Then there exists $f \in C(K)$ such that*

$$\int_K f(t)\,dt = 0 \quad \textit{and} \quad \|\chi + g\|_2 \leqslant \varepsilon, \quad \textit{where} \quad g(x) = \int_{K\cap[x,1]} f(t)\,dt$$

and χ is the indicator function of K ($\chi(x) = 1$ if $x \in K$ and $\chi(x) = 0$ if $x \in [0,1] \setminus K$).

Proof. If the Lebesgue measure $\mu(K)$ of K is zero, the statement is trivially true since the function $f \equiv 0$ satisfies the desired conditions for any $\varepsilon > 0$. Thus, we can assume that $\mu(K) > 0$. Let $n \in \mathbb{N}$. Since K is nowhere dense and has positive Lebesgue measure, we can choose $n \in \mathbb{N}$ and $\alpha_k, \beta_k, a_k, b_k, u_k, v_k \in [0,1]\setminus K$ for $1 \leqslant k \leqslant n$ in such a way that

$$\alpha_k < \beta_k < a_k < b_k < u_k < v_k \ \text{ for } \ 1 \leqslant k \leqslant n \ \text{ and } \ v_{k-1} < \alpha_k \ \text{ for } \ 2 \leqslant k \leqslant n,$$

$$0 < \mu(K \cap [\alpha_k, \beta_k]) < \tfrac{\varepsilon^2}{16n} \ \text{ and } \ 0 < \mu(K \cap [u_k, v_k]) < \tfrac{\varepsilon^2}{16n} \ \text{ for } \ 1 \leqslant k \leqslant n, \tag{4.1}$$

$$\mu\Big(\Big(\bigcup_{k=1}^n [\alpha_k, v_k]\Big) \setminus K\Big) < \tfrac{\varepsilon^2}{8}. \tag{4.2}$$

Consider the function $f : K \to \mathbb{R}$ defined by the formula

$$f(x) = \begin{cases} \frac{1}{\mu(K\cap[\alpha_k,\beta_k])} & \text{if } x \in K \cap [\alpha_k, \beta_k],\ 1 \leqslant k \leqslant n; \\ \frac{-1}{\mu(K\cap[u_k,v_k])} & \text{if } x \in K \cap [u_k, v_k],\ 1 \leqslant k \leqslant n; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $f \in C(K)$ and

$$\int_K f(t)\,dt = \sum_{k=1}^n \Big(\int_{K\cap[\alpha_k,\beta_k]} f(t)\,dt - \int_{K\cap[u_k,v_k]} f(t)\,dt\Big) = \sum_{k=1}^n (1-1) = 0.$$

Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \int_{K\cap[x,1]} f(t)\,dt.$$

From the definition of f it follows that $|g(x)| \leqslant 1$ for any $x \in [0,1]$, $g(x) = -1$ if $x \in \bigcup\limits_{k=1}^n [\beta_k, u_k]$ and $\chi(x) = g(x) = 0$ if $x \in [0,1] \setminus \bigcup\limits_{k=1}^n [\alpha_k, v_k]$. Hence the set $\Omega = \{x \in [0,1] : g(x) + \chi(x) \neq 0\}$ is contained in the union

$$\Omega_1 = \left(\left(\bigcup_{k=1}^n [\alpha_k, v_k]\right) \setminus K\right) \cup \left(\bigcup_{k=1}^n ([\alpha_k, \beta_k] \cap K)\right) \cup \left(\bigcup_{k=1}^n ([u_k, v_k] \cap K)\right).$$

Therefore

$$\|g+\chi\|_2^2 = \int_0^1 (g(x) + \chi(x))^2 \leqslant 4\mu(\Omega) \leqslant 4\mu(\Omega_1).$$

Using (4.1) and (4.2), we see that $\mu(\Omega_1) \leqslant \varepsilon^2/4$. Hence $\|g+\chi\|_2 \leqslant \varepsilon$. □

Lemma 4.4. *Let $\{e_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis in a separable Hilbert space H and scalar sequences $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ be such that*

$$\sum_{n=1}^{\infty}(|\gamma_n|^2+|\delta_n|^2)<\infty. \tag{4.3}$$

Let also $f_0=e_0+\sum\limits_{n=1}^{\infty}\gamma_n e_n$ and $f_n=e_n-\delta_n e_0$ for $n\in\mathbb{N}$. Then the linear span of $\{f_n:n\in\mathbb{Z}_+\}$ is dense in H if and only if

$$\sum_{n=1}^{\infty}\gamma_n\delta_n\neq -1. \tag{4.4}$$

Proof. Condition (4.3) implies that the linear operator $T:H\to H$ such that $Te_0=\sum\limits_{n=1}^{\infty}\gamma_n e_n$ and $Te_n=-\delta_n e_0$ for $n\in\mathbb{N}$ is bounded. Since the range of T is at most two-dimensional, T is compact. By the Fredholm theorem [3], the operator $S=I+T$ has dense range if and only if S is injective. Since $Se_n=f_n$ for $n\in\mathbb{Z}_+$, the linear span of $\{f_n\}_{n\in\mathbb{Z}_+}$ is dense in H if and only if the operator S injective.

The equation $Sx=0$, $x\in H$ can be rewritten as

$$\langle x,e_0\rangle\left(1+\sum_{n=1}^{\infty}\gamma_n\delta_n\right)=0 \ \text{ and } \ \langle x,e_n\rangle=\gamma_n\langle x,e_0\rangle \ \text{ for any } \ n\in\mathbb{N}.$$

If $\sum\limits_{n=1}^{\infty}\gamma_n\delta_n\neq -1$, the first equation implies $\langle x,e_0\rangle=0$ and the rest yield $\langle x,e_n\rangle=0$ for each $n\in\mathbb{N}$. Thus in this case $x=0$. That is, S is injective and therefore the linear span of $\{f_n:n\in\mathbb{Z}_+\}$ is dense in H. If $\sum\limits_{n=1}^{\infty}\gamma_n\delta_n=-1$, the system of the equations in the above display has the non-zero solution $x=x_0+\sum\limits_{n=1}^{\infty}\gamma_n e_n\in H$. Hence S is not injective and therefore the linear span of $\{f_n:n\in\mathbb{Z}_+\}$ is non-dense. □

We are ready to prove Theorem 1.7. First, if $n\in\mathbb{N}$ and H is n-dimensional, then $W^{1,2}([0,1],H)$ is isomorphic to the free $W^{1,2}[0,1]$-module with n generators and the nicety of $W^{1,2}([0,1],H)$ follows from Proposition 1.5. It is easy to see that a direct (module) summand of a nice module is nice. Thus the proof of Theorem 1.7 will be complete if we verify that $W^{1,2}([0,1],\ell_2)$ is non-nice. In order to do this, we have to construct a proper closed $W^{1,2}[0,1]$-submodule M of $W^{1,2}([0,1],\ell_2)$ such that none of the characters on $W^{1,2}([0,1],\ell_2)$ vanishes on M. Now we shall do just that.

Pick a nowhere dense compact set $K\subset[0,1]$ of positive Lebesgue measure and let χ be the indicator function of K. By Lemma 4.3, there exists $A_n\in C(K)$ such that for any $n\in\mathbb{N}$,

$$\int\limits_K A_n(x)\,dx=0, \tag{4.5}$$

$$\|B_n+\chi\|_2<2^{-n}, \quad \text{where} \quad B_n(x)=\int\limits_{K\cap[x,1]}A(t)\,dt. \tag{4.6}$$

We also set $A_0=0$, $B_0=0$ and $S_0=1$. By Lemma 4.2, there exist $S_n\in C^1[0,1]$ such that

$$S_n'\big|_K=A_n \ \text{ and } \ \|S_n-1\|_\infty<2^{-n} \ \text{ for each } n\in\mathbb{N}. \tag{4.7}$$

Denote $\rho_n=n^2(S_n-S_{n-1})$ for $n\in\mathbb{N}$. Then $\rho_n\in C^1[0,1]$ and according to (4.7),

$$\|\rho_n\|_\infty\leqslant n^2(\|S_n-1\|_\infty+\|S_{n-1}-1\|_\infty)\leqslant n^2(2^{1-n}+2^{-n})=3n^2 2^{-n} \ \text{ for each } n\in\mathbb{N}. \tag{4.8}$$

Let also $\{e_n\}_{n\in\mathbb{Z}_+}$ be the standard orthonormal basis in ℓ_2. Consider the functions $f^{[n]} \in W_2^1([0,1],\ell_2)$ defined by the formulas

$$f^{[0]}(x) = e_0 + \sum_{n=1}^{\infty} n^{-2} e_n \quad \text{and} \quad f^{[n]}(x) = e_n - \rho_n(x)e_0 \quad \text{for } n \in \mathbb{N}.$$

Let now M be the closed $W^{1,2}[0,1]$-submodule of $W^{1,2}([0,1],\ell_2)$ generated by the set $\{f^{[n]} : n \in \mathbb{Z}_+\}$. Equivalently, M is the closed linear span in $W^{1,2}([0,1],\ell_2)$ of the set $\{\varphi f^{[n]} : n \in \mathbb{Z}_+,\ \varphi \in W^{1,2}[0,1]\}$.

It is easy to see that every character on $W^{1,2}([0,1],\ell_2)$ has the shape

$$\varphi_{t,y}(f) = \langle f(t), y\rangle_H, \quad \text{where } t \in [0,1] \text{ and } y \in \ell_2 \setminus \{0\}.$$

Thus in order for every character on $W^{1,2}([0,1],\ell_2)$ not to vanish on M it is necessary and sufficient for $M_t = \{f(t) : f \in M\}$ to be dense in ℓ_2 for every $t \in [0,1]$. Let $t \in [0,1]$. By definition of ρ_n and (4.7), we have

$$\sum_{n=1}^{\infty} n^{-2}\rho_n(t) = \lim_{m\to\infty} \sum_{n=1}^{m} (S_n(t) - S_{n-1}(t)) = \lim_{m\to\infty} (S_m(t) - S_0(t)) = 0 \neq -1. \tag{4.9}$$

By Lemma 4.4 with $\gamma_n = n^{-2}$ and $\delta_n = \rho_n(t)$, the linear span of $\{f^{[n]}(t)\}_{n\in\mathbb{Z}_+}$ is dense in ℓ_2. Since $f^{[n]} \in M$, M_t is dense in ℓ_2. Thus none of the characters on $W^{1,2}([0,1],\ell_2)$ vanishes on M. It remains to verify that $M \neq W^{1,2}([0,1],\ell_2)$. Consider $g_n \in W_2^1[0,1]^*$ for $n \in \mathbb{Z}_+$, defined by the formula

$$g_n(\varphi) = \int_K (\rho_n\varphi)'(x)\,dx, \quad \text{where } \rho_0 \text{ is assumed to be identically } 1.$$

We start with estimating the norms of the functionals g_n. Clearly,

$$g_n(\varphi) = \int_K \rho_n(x)\varphi'(x)\,dx + \int_K \rho_n'(x)\varphi(x)\,dx \quad \text{for any } \varphi \in W^{1,2}[0,1]. \tag{4.10}$$

Since $\rho_n'(x) = n^2(S_n'(x) - S_{n-1}'(x)) = n^2(A_n(x) - A_{n-1}(x))$ for $x \in K$, we have

$$\int_K \rho_n'(x)\varphi(x)\,dx = n^2 \int_K (A_n(x) - A_{n-1}(x))\varphi(x)\,dx = n^2 \int_0^1 (B_{n-1}'(x) - B_n'(x))\varphi(x)\,dx.$$

By (4.5) and (4.6),

$$B_n(0) = B_n(1) = 0 \quad \text{for } n \in \mathbb{Z}_+.$$

Integrating by parts and using the above display, we obtain

$$\int_K \rho_n'(x)\varphi(x)\,dx = n^2 \int_0^1 (B_{n-1}'(x) - B_n'(x))\varphi(x)\,dx = n^2 \int_0^1 (B_n(x) - B_{n-1}(x))\varphi'(x)\,dx.$$

This formula together with (4.10) yields

$$|g_n(\varphi)| \leqslant \|\varphi'\|_2(\|\rho_n\|_2 + n^2\|B_n - B_{n-1}\|_2) \quad \text{for } n \in \mathbb{N}.$$

Since $\|\rho_n\|_2 \leqslant 3n^2 2^{-n}$ and $\|B_n - B_{n-1}\|_2 \leqslant \|B_n + \chi\|_2 + \|B_{n-1} + \chi\|_2 \leqslant 2^{1-n} + 2^{-n} = 3\cdot 2^{-n}$, we have $|g_n(\varphi)| \leqslant 6n^2 2^{-n}\|\varphi\|_{1,2}$. Hence $\|g_n\| \leqslant 6n^2 2^{-n}$ for each $n \in \mathbb{N}$. Therefore $\sum\limits_{n=0}^{\infty} \|g_n\|^2 < \infty$. Thus the formula

$$g(h) = \sum_{n=0}^{\infty} g_n(h_n)$$

defines a continuous linear functional on $W^{1,2}([0,1],\ell_2)$, where, as usual, $h_n(t) = \langle h(t), e_n\rangle$. Since $g_0 \neq 0$, we have $g \neq 0$. In order to show that $M \neq W^{1,2}([0,1],\ell_2)$, it suffices to verify that $g(h) = 0$ for any $h \in M$.

For this it is enough to check that $g(\varphi f^{[n]}) = 0$ for every $\varphi \in W^{1,2}[0,1]$ and $n \in \mathbb{Z}_+$. First, let $n \in \mathbb{N}$. Then by definition of g_n, we immediately have

$$g(\varphi f^{[n]}) = g_n(\varphi) - g_0(\rho_n \varphi) = 0.$$

It remains to prove that $g(\varphi f^{[0]}) = 0$. Using the uniform convergence of the series $\sum\limits_{n=1}^{\infty} n^{-2}\rho_n$ provided by the estimate (4.8), we have

$$g(\varphi f^{[0]}) = g_0(\varphi) + \sum_{n=1}^{\infty} n^{-2} g_n(\varphi) = \int_K \Big(\varphi'(x) + \sum_{n=1}^{\infty} n^{-2}(\rho_n\varphi)'(x)\Big)\,dx =$$
$$= \int_K \varphi'(x)\Big(1 + \sum_{n=1}^{\infty} n^{-2}\rho_n(x)\Big)\,dx + \lim_{m\to\infty}\int_K \varphi(x)\Big(\sum_{n=1}^{m} n^{-2}\rho_n'(x)\Big)\,dx.$$

By (4.9), $\sum\limits_{n=1}^{\infty} n^{-2}\rho_n(x) \equiv 0$. On the other hand, using (4.7) and the equality $S_0 = 1$, we have

$$\sum_{n=1}^{m} n^{-2}\rho_n'(x) = \sum_{n=1}^{m}(S_n'(x) - S_{n-1}'(x)) = S_m'(x) = A_m(x) \ \text{ for each } \ x \in K.$$

Hence

$$g(\varphi f^{[0]}) = \int_K \varphi'(x)\,dx + \lim_{m\to\infty}\int_K \varphi(x)A_m(x)\,dx. \tag{4.11}$$

Integrating by parts, we obtain

$$\int_K \varphi(x)A_m(x)\,dx = -\int_0^1 \varphi(x)B_m'(x)\,dx = \int_0^1 \varphi'(x)B_m(x)\,dx = \int_0^1 \varphi'(x)(B_m(x)+\chi(x))\,dx - \int_K \varphi'(x)\,dx.$$

According to (4.11) and the above display,

$$g(\varphi f^{[0]}) = \lim_{m\to\infty}\int_0^1 \varphi'(x)(B_m(x)+\chi(x))\,dx. \tag{4.12}$$

By (4.6) and (4.12), $g(\varphi f^{[0]}) = 0$ for each $\varphi \in W^{1,2}[0,1]$. Thus $g(h) = 0$ for every $h \in M$ and therefore $M \neq W^{1,2}([0,1],\ell_2)$. The proof of Theorem 1.7 is complete.

5 Remarks

One can easily generalize Theorem 1.7 by taking most any algebra of smooth functions instead of $W^{1,2}[0,1]$. For example, following the same route of argument with few appropriate amendments one can show that if X is an infinite dimensional Banach space and $k \in \mathbb{N}$, then $C^k([0,1],X)$ as a $C^k[0,1]$-module is non-nice. We opted for $W^{1,2}([0,1],H)$ to make a point that even the friendly Hilbert space environment does not save the day.

Theorem 1.7 says that there are weird proper closed submodules of $W^{1,2}([0,1],\ell_2)$ which are not contained in any closed submodule of codimension 1. The following question remains wide open.

Question 5.1. *Characterize closed submodules of* $W^{1,2}([0,1],\ell_2)$.

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